In a nutshell: Cubic splines

Given two points (t_{k-1}, y_{k-1}) and (t_k, y_k) where $h = t_k - t_{k-1}$, given no additional information, the best approximation of a point between these two is to use a linear interpolation polynomial, where we approximate the value at $t_{k-1} + \delta h$ is

$$y_{k-1} + \delta(y_k - y_{k-1})$$

If, however, we also know the slopes $y_{k-1}^{(1)}$ and $y_k^{(1)}$ at the points at t_{k-1} and t_k, respectively, we can find the cubic polynomial that matches both the y-values and the slopes at both of these end points:

$$\left\{\left[\left(h\left(y_{k-1}^{(1)} + y_k^{(1)}\right) + 2\left(y_{k-1} - y_k\right)\right)\delta - \left(h\left(2y_{k-1}^{(1)} + y_k^{(1)}\right) + 3\left(y_{k-1} - y_k\right)\right)\right]\delta + hy_{k-1}^{(1)}\right\}\delta + y_{k-1}.$$

This is displayed using Horner's rule and it is assumed that $0 < \delta < 1$.

Derivation

This is found by solving the system of linear equations defined by

$$\begin{aligned} a_3t_{k-1}^3 + a_2t_{k-1}^2 + a_1t_{k-1} + a_0 &= y_{k-1} \\ a_3t_k^3 + a_2t_k^2 + a_1t_k + a_0 &= y_k \\ 3a_3t_{k-1}^2 + 2a_2t_{k-1} + a_1 &= y_{k-1}^{(1)} \\ 3a_3t_k^2 + 2a_2t_k + a_1 &= y_k^{(1)} \end{aligned}$$

or

$$\left(\begin{array}{cccc:c} t_{k-1}^3 & t_{k-1}^2 & t_{k-1} & 1 & y_{k-1} \\ t_k^3 & t_k^2 & t_k & 1 & y_k \\ 3t_{k-1}^2 & 2t_{k-1} & 1 & 0 & y_{k-1}^{(1)} \\ 3t_k^2 & 2t_k & 1 & 0 & y_k^{(1)} \end{array}\right)$$

where we have the polynomial $a_3t^3 + a_2t^2 + a_1t + a_0$ and its derivative $3a_3t^2 + 2a_2t + a_1$. This solution, however, has a higher condition number, and therefore to reduce this, we shift and scaling the t-values to 0 and 1, respectively, resulting in the simpler system of linear equations where the solution is in terms of δ and not t, as given above:

$$\left(\begin{array}{cccc:c} 0 & 0 & 0 & 1 & y_{k-1} \\ 1 & 1 & 1 & 1 & y_k \\ 0 & 0 & 1 & 0 & hy_{k-1}^{(1)} \\ 3 & 2 & 1 & 0 & hy_k^{(1)} \end{array}\right)$$